JOURNAL OF NANO- AND ELECTRONIC PHYSICS
Vol. **10** No 6, 06015(7pp) (2018)

ЖУРНАЛ НАНО- ТА ЕЛЕКТРОННОЇ ФІЗИКИ
Том **10** № 6, 06015(7cc) (2018)

Extended of the Schrödinger Equation with New Coulomb Potentials plus Linear and Harmonic Radial Terms in the Symmetries of Noncommutative Quantum Mechanics

Abdelmadjid Maireche*

Laboratory of Physics and Material Chemistry, Physics Department, Sciences Faculty, University of M'sila-M'sila Algeria

(Received 08 July 2018; revised manuscript received 28 November 2018; published online 18 December 2018)

In this recently work, we have obtained the analytical solutions of the modified Schrödinger equation (MSE) with new Coulomb potentials plus linear and harmonic radial terms (*NCPLHR*T) for hydrogenic atoms in noncommutative 3-dimensional real space-phase (NC: 3D-RSP). We applied, the generalized Bopp's shift method and standard perturbation theory in the framework of two infinitesimal parameters $\left(\Theta,\bar{\theta}\right)$ due to (space-phase) noncommutativity, we obtained new energy eigenvalues $E_{\text{nc -(u-d)clh}}\left(n,\lambda,g,Z,j,l,s,m\right)$, which depended with discrete atomic quantum numbers $\left(j,l,s,m\right)$ and parameters of studied potential $\left(\lambda,g,Z\right)$, in addition to the corresponding new Hamiltonian operator $H_{nc-chl}\left(\hat{p}_i,\hat{x}_i\right)$. Our research also conveys another innovative feature, the global group symmetry (NC: 3D-RSP) were broken simultaneously and replaced by the residual local subgroup (NC: 3D-RS) under interaction of hydrogenic atoms with *NCPLHR*T.

Keywords: Schrödinger equation, Coulomb potentials plus linear and harmonic radial terms, Noncommutative space-phase, Star product and Generalized Bopp's shift method.

DOI: 10.21272/jnep.10(6).06015 PACS numbers: 11.10.Nx, 32.30 – r, 03.65 – w

1. INTRODUCTION

Generally, throughout the literature, the Coulomb potentials plus linear and harmonic radial terms (CPLHRT) of the form $V\left(r\right)=-Z/r+gr+\lambda r^2$ have attracted great interest in atomic and molecular physics and quantum chromodynamics, this potentials supposed to be responsible for the interaction between quark and antiquark and alternatively, it is called Killingbeck or Cornell plus harmonic potential [1-3]. The aim of our work is to extended, the study of Enrique Castro and Pablo Martin [1] to the case of extended quantum mechanics (EQM), or noncommutative quantum mechanics (NCQM) to finding other new applications and more profound interpretations in the subatomic scales. On the other hand, we extend our study of [4] from noncommutative two dimensional real spaces (NC: 2D-RS) to the case of three-dimensional real space-phase (NC: 3D-RSP). We based on previous studies of other authors and some of our related works in this context. The EQM known by noncommutativity of space-time, introduced firstly by Heisenberg, and formalized by Snyder at 1947, suggest by the physical recent results in string theory. The nonrelativistic energy levels for hydrogenic atoms, which interacted with new Coulomb potentials plus linear and harmonic radial terms (*NCPLHR*T) in the context of NC space, have not been obtained yet. In last few years many effort has been produced to study some potentials using the notions of noncommutativity of space and phase based essentially on Seiberg-Witten map and generalized Bopp's shift method and the star product, defined on the first order of two infinitesimal parameters antisymmetric $2\left(\theta^{\mu\nu},\bar{\theta}^{\mu\nu}\right)\equiv\varepsilon^{\mu\nu\rho}\left(\theta_{\rho},\bar{\theta}_{\rho}\right)$ as [5-9]:

$$\begin{aligned}&\left(f*g\right)\left(x,p\right)=\left(fg\right)\left(x,p\right)-\\&-\frac{i}{2}\left(\theta^{\mu\nu}\partial_{\mu}^{x}f\,\partial_{\nu}^{x}g+\bar{\theta}^{\mu\nu}\partial_{\mu}^{p}f\partial_{\nu}^{p}g\right)\left(x,p\right)\end{aligned}\tag{1}$$

As direct results for the above two modes of star product due to the space-space and phase-phase noncommutativity, allow us to finding new none nulls commutators [10-14]:

$$\left[x^{\mu},x^{\nu}\right]_{*}=i\theta^{\mu\nu}\text{ and }\left[\hat{p}^{\mu},\hat{p}^{\nu}\right]_{*}=i\bar{\theta}^{\mu\nu}\tag{2}$$

The present paper is arranged as follows. Section 2 is a brief outline of the ordinary SE with *NCPLHR*T on based to the main ref. [1]. The Section 3 is devoted to studying the MSE by applying the generalized Bopp's shift method and standard perturbation theory we find the quantum spectrum of n^{th} excited levels for modified spin-orbital interaction in the framework of the global quantum group (NC-3D: RSP) for *NCPLHR*T, then we end this section by derive the magnetic spectrum for *NCPLHR*T. In the fourth section, we resume the global spectrum and corresponding NC Hamiltonian operator for *NCPLHR*T and corresponding energy levels of hydrogenic atoms. Conclusion of the work is placed in the last section.

2. REVIEW THE EIGENVALUES OF THE SCHRÖDINGER EQUATION WITH *CPLHRT*

Let us begin this section by reviewing the nonrelativ-

* abmaireche@gmail.com

istic quantum description of an atom or molecule with an induced by spherically symmetric potential that has the form [1]:

$$V(r) = -Z/r + gr + \lambda r^2 \tag{3}$$

The parameters Z, g ($E_h e^{-1} a_0^{-1}$) and λ are atomic number, the electric field (in atomic units) and a positive constant, respectively, The second term in the above equation corresponds to the scalar potential proportional to the radial distance. The Hamiltonian operator for hydrogenic atoms with a polynomial perturbation is [1]:

$$\hat{H}(p_i, x_i) = H_0(r) + V_1(r) \tag{4}$$

The principal Hamiltonian operator $H_0(r)$ and the perturbed potential $V_1(r)$ as [1]:

$$\begin{aligned} H_0(r) &= -\frac{1}{2}\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \lambda r^2 + \frac{\vec{L}^2}{2r^2} \\ V_1(r) &= -Z/r + gr \end{aligned} \tag{5}$$

here $\vec{L}^2$ is the angular momentum operator. The complex eigenfunctions $\Psi_{nlm}(r,\theta,\phi) = R_{nl}(r) Y_l^m(\theta,\phi)$ in 3-dimensional space while the radial part $R_{nl}(r)$ satisfies (Here we use atomic units $\hbar = m_e = c = 1$) [1]:

$$\begin{aligned} &\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + 2\left(E_{nl} + Z/r - gr - \lambda r^2\right)\right] \times \\ &\times R_{nl}(r) = 0 \end{aligned} \tag{6}$$

Where l and E_{nl} represent angular momentum and the energy while $-l \le m \le +l$. The eigenvalues and eigenfunctions of the unperturbed states are, respectively as [1]:

$$\begin{aligned} &\Psi_{nlm}(r,\theta,\phi) = N r^l \exp(-\lambda r^2/2) {}_1F_1 \times \\ &\times \left(-n, l+3/2; \lambda r^2\right) Y_l^m(\theta,\phi) \end{aligned} \tag{7}$$

where N is the normalization constant and $Y_l^m(\theta,\phi)$ are the spherical harmonics. The eigenvalues corresponding the global potential were obtained in [1] as:

$$E_{n,l} = \sqrt{2\lambda}\left(2n + l + 3/2\right) + E_{nl}^{(RS)} \tag{8}$$

while the perturbed energy $E_{nl}^{(RS)}$ corresponded the perturbed potential $V_1(r)$ obtained after applying the HVT and HFT method [1]:

$$E_{nl}^{(RS)} = \sum_{j=0}^{\infty}\left(b_j^{(0)} \lambda^{j+1/2} + b_j^{(1)} \lambda^{j+1/4} + b_j^{(2)} \lambda^{j} + b_j^{(3)} \lambda^{-(j+1/4)}\right) \tag{9}$$

On the other hand, using the following relationship between the associated Laguerre function $L_n^{l+1/2}(\lambda r^2)$ and the hypergeometric function ${}_1F_1(-n, l+3/2; \lambda r^2)$ [15]:

$${}_1F_1\left(-n, l+3/2; \lambda r^2\right) = \frac{n!\Gamma(l+3/2)}{\Gamma(l+n+3/2)} L_n^{l+1/2}\left(\lambda r^2\right) \tag{10}$$

we may further rewrite the wave function for *CPLHrt* as :

$$\Psi_{nlm}(r,\theta,\phi) = N\frac{n!\Gamma(l+3/2)}{\Gamma(l+n+3/2)} r^l \exp(-\lambda r^2/2) L_n^{l+1/2}\left(\lambda r^2\right) Y_l^m(\theta,\phi) \tag{11}$$

The purpose of the present paper is to attempt study the MSE with *NCPLHR*T potential (see below) in (NC: 3D-RSP) symmetries using the generalized Bopp's shift method which depend on the concepts that we present below in the third section to discover the new symmetries and a possibility to obtain another applications to this potential in different fields.

3. METHOD AND THEORETICAL APPROACH

In this section, we shall give an overview or a brief preliminary for *NCPLHR*T, in (NC: 3D-RSP) symmetries. To perform this task the physical form of modified Schrödinger equation (MSE), it is necessary to replace ordinary three-dimensional Hamiltonian operators $\hat{H}(p_i, x_i)$, ordinary complex wave function $\Psi\left(\vec{r}\right)$ and ordinary energy E_{nl} by new three Hamiltonian operators $\hat{H}_{nc-clh}(\hat{p}_i, \hat{x}_i)$, new complex wave function $\hat{\Psi}\left(\vec{\hat{r}}\right)$ and new values E_{nc-clh}, respectively. In addition to replace the ordinary old product by new star product $(*)$, which allow us to constructing the MSE in (NC-3D: RSP) symmetries as [16-19]:

$$\hat{H}_{clh}\left(p_i, x_i\right)\Psi\left(\vec{r}\right) = E_{nl}\Psi\left(\vec{r}\right) \Rightarrow \hat{H}\left(\hat{p}_i, \hat{x}_i\right) * \Psi\left(\vec{\hat{r}}\right) = E_{nc-clh}\Psi\left(\vec{\hat{r}}\right) \tag{12}$$

The Bopp's shift method employed in the solutions enables us to explore an effective way of obtaining the *NCPLHR*T in EQM, it based on the following new commutators: [16-19]:

$$\begin{aligned} \left[x_i, x_j\right] = 0 \to \left[\hat{x}_i, \hat{x}_j\right] = \left[\hat{x}_i(t), \hat{x}_j(t)\right] = i\theta_{ij} \\ \left[p_i, p_j\right] = 0 \to \left[\hat{p}_i, \hat{p}_j\right] = \left[\hat{p}_i(t), \hat{p}_j(t)\right] = i\bar{\theta}_{ij} \end{aligned} \tag{13}$$

The new generalized positions and momentum coordinates $(\hat{x}_i,\hat{p}_i)$ in (NC: 3D-RSP) symmetries are depended with corresponding usual generalized positions and momentum coordinates (x_i,p_i) in ordinary quantum mechanics by the following, respectively [16-19]:

$$\left(x_i,p_i\right)\Rightarrow\left(\hat{x}_i,\hat{p}_i\right)=\left(x_i-\frac{\theta_{ij}}{2}p_j,p_i+\frac{\bar{\theta}_{ij}}{2}x_j\right) \tag{14}$$

The above equation allows us to obtain the two operators ($\hat{r}^2$ and $\hat{p}^2$) in (NC-3D: RSP), respectively [20-22]:

$$\left(r^2,p^2\right)\Rightarrow\left(\hat{r}^2,\hat{p}^2\right)=\left(r^2-\mathbf{L}\vec{\Theta}\ ,\ p^2\ +\ \vec{\mathbf{L}}\bar{\vec{\theta}}\right) \tag{15}$$

The two couplings $\mathbf{L}\Theta$ and $\vec{\mathbf{L}}\bar{\vec{\theta}}$ are $\left(L_x\Theta_{12}+L_y\Theta_{23}+L_z\Theta_{13}\right)$ and $\left(L_x\bar{\theta}_{12}+L_y\bar{\theta}_{23}+L_z\bar{\theta}_{13}\right)$, respectively and ($L_x$, L_y and L_z) are the three components of angular momentum operator $\vec{L}$ while the new parameter Θ_{ij} equal $\theta_{ij}/2$. Thus, the reduced Schrödinger equation (without star product) can be written as:

$$\begin{gathered}\hat{H}\left(\hat{p}_i,\hat{x}_i\right)*\Psi\left(\vec{\hat{r}}\right)=E_{nc-clh}\Psi\left(\vec{\hat{r}}\right)\\ \Rightarrow H_{nc-clh}\left(\hat{p}_i,\hat{x}_i\right)\psi\left(\vec{r}\right)=E_{nc-clh}\psi\left(\vec{r}\right)\end{gathered} \tag{16}$$

The new operator of Hamiltonian $H_{nc-clh}\left(\hat{p}_i,\hat{x}_i\right)$ can be expressed as:

$$H_{nc-clh}\left(\hat{p}_i,\hat{x}_i\right)\equiv\frac{\hat{p}^2}{2}+V_{clh}\left(\hat{r}\right) \tag{17}$$

The *NCPLHRT* $V_{clh}\left(\hat{r}\right)$ is given by:

$$V_{clh}\left(\hat{r}\right)=-Z/\hat{r}+g\hat{r}+\lambda\hat{r}^2 \tag{18}$$

After straightforward calculations, we can obtain the important terms ($-Z/\hat{r}$, $g\hat{r}$ and $\lambda\hat{r}^2$), which will be use to determine the *NCPLHR*T in (NC: 3D- RSP) symmetries as:

$$\begin{gathered}-\frac{Z}{r}\to-\frac{Z}{\hat{r}}=-\frac{Z}{r}-\frac{Z\vec{\mathbf{L}}\vec{\Theta}}{2r^3}+O(\Theta)\\ gr\to g\hat{r}=g\hat{r}=gr-\frac{g}{2r}\vec{\mathbf{L}}\vec{\Theta}+O(\Theta)\\ \lambda r^2\to\lambda\hat{r}^2=\lambda r^2-\lambda\vec{L}\vec{\Theta}+O(\Theta)\end{gathered} \tag{19}$$

We further the equations (19) and (15) into Equation (17) we obtained the global our working new Hamiltonian operator $H_{\text{nc-clh}}\left(\hat{r}\right)$ for *NCPLHR*T satisfies the equation in (NC: 3D-RSP) symmetries:

$$H_{nc-\text{clh}}\ \left(\hat{r}\hat{p}\ \right)=H_{\text{clh}}\left(p_i,x_i\right)-\left(\frac{Z}{2r^3}+\frac{g}{2r}+\lambda\right)\vec{\mathbf{L}}\vec{\Theta}+\frac{\vec{\mathbf{L}}\bar{\vec{\theta}}}{2} \tag{20}$$

where the operator $H_{\text{clh}}\left(p_i,x_i\right)$ is just the ordinary Hamiltonian operator for *CPLHRT* in commutative space:

$$H_{\text{clh}}\left(p_i,x_i\right)=\frac{p^2}{2}-Z/r+gr+\lambda r^2 \tag{21}$$

while the rest four terms are proportional's with two infinitesimals parameters (Θ and $\bar{\theta}$) and then we can considered as a perturbations terms $H_{\text{per-clh}}\left(r\right)$ in (NC: 3D-RSP) symmetries for *NCPLHRT* as:

$$H_{\text{per-clh}}\left(r\right)=-\left(\frac{Z}{2r^3}+\frac{g}{2r}+\lambda\right)\vec{\mathbf{L}}\vec{\Theta}+\frac{\vec{\mathbf{L}}\bar{\vec{\theta}}}{2} \tag{22}$$

3.1 The Exact Modified Spin-Orbital Spectrum for *NCPLHRT* in Global (NC: 3D- RSP) Symmetries

In this subsection, we apply the same strategy, which we have seen in our previously works [19-22], under such particular choice, one can easily reproduce both couplings ($\vec{\mathbf{L}}\vec{\Theta}$ and $\vec{\mathbf{L}}\bar{\vec{\theta}}$) to the new physical forms ($\gamma\Theta\vec{L}\vec{S}$ and $\gamma\bar{\theta}\vec{L}\vec{S}$), respectively, to obtain the new forms of $H_{\text{so-clh}}\left(r,\Theta,\bar{\theta}\right)$ for 3D-*NCPLHRT* as follows:

$$H_{\text{so-chl}}\left(r,\Theta,\bar{\theta}\right)\equiv-\gamma\left\{\left(\frac{Z}{2r^3}+\frac{g}{2r}+\lambda\right)\Theta-\frac{\bar{\theta}}{2}\right\}\vec{L}\vec{S} \tag{23}$$

Here $\gamma\approx\frac{1}{137}$ is a new constant, which play the role of fine structure constant, we have chosen the two vectors ($\vec{\Theta}$ and $\bar{\vec{\theta}}$) parallel to the spin $\vec{S}$ of hydrogenic atoms. Furthermore, the above perturbative terms $H_{\text{per-chl}}\left(r\right)$ can be rewritten to the following new form:

$$H_{so-\text{chl}}\left(r,\Theta,\bar{\theta}\right)=-\frac{\gamma}{2}\left\{\left(\frac{Z}{2r^3}+\frac{g}{2r}+\lambda\right)\Theta-\frac{\bar{\theta}}{2}\right\}G^2 \tag{24}$$

With $G^2\equiv\vec{J}-\vec{L}-\vec{S}^2$, this operator traduces the coupling between spin $\vec{S}$ and orbital momentum $\vec{L}\vec{S}$. The set ($H_{so-chl}\left(r,\Theta,\bar{\theta}\right)$, J^2, L^2, S^2 and J_z) forms a complete of conserved physics quantities and for $\vec{S}=1\vec{/}2$, the eigenvalues of the spin orbital coupling operator are $k_\pm\equiv\frac{1}{2}\left\{\left(l\pm\frac{1}{2}\right)(l\pm\frac{1}{2}+1)+l(l+1)-\frac{3}{4}\right\}$ corresponding: $j=l+1/2$ (spin up) and $j=l-1/2$ (spin down), respectively, then, one can form a diagonal (3×3) matrix, with diagonal elements are $\left(H_{so-\text{clh}}\right)_{11}$, $\left(H_{so-\text{clh}}\right)_{22}$ and $\left(H_{so-\text{clh}}\right)_{33}$ for *NCPLHRT* in (NC: 3D-RSP) symmetries, as:

After profound calculation, one can show that, the new radial function $R_{nl}(r)$ satisfying the following differential equation for *NCPLHRT* in the symmetries of (NC: 3D- RSP):

$$\begin{aligned}&(H_{so-\mathrm{clh}})_{11}=-\gamma k_+\left\{\left(\frac{Z}{2r^3}+\frac{g}{2r}+\lambda\right)\Theta-\frac{\bar{\theta}}{2}\right\}\\ &\text{if } j=l+1/2\\ &(H_{so-\mathrm{clh}})_{22}=-\gamma k_-\left\{\left(\frac{Z}{2r^3}+\frac{g}{2r}+\lambda\right)\Theta-\frac{\bar{\theta}}{2}\right\}\\ &\text{if } j=l-1/2\\ &(H_{so-\mathrm{clh}})_{33}=0\end{aligned} \tag{25}$$

$$\frac{d^2R_{nl}(r)}{dr^2}+2\left[E_{nl}-Z/r-gr-\lambda r^2-\frac{2l(l+1)}{2r^2}+\left(\frac{Z}{2r^3}+\frac{g}{2r}+\lambda\right)\bar{\mathbf{L}}\Theta-\frac{\bar{\mathbf{L}}\vec{\bar{\theta}}}{2}\right]R_{nl}(r)=0 \tag{26}$$

The two terms which composed the expression of $H_{\text{per-clh}}(r)$ are proportional with two infinitesimals parameters (Θ and $\bar{\theta}$), thus, in what follows, we proceed to solve the modified radial part of the MSE that is, equation (21) by applying standard perturbation theory for their exact solutions at first order of two parameters Θ and $\bar{\theta}$.

3.2 The Exact Modified Spin-Orbital Spectrum of Hydrogenic Atoms Under *NCPLHRT* Interactions and Spontaneous Symmetry Breaking of (NC: 3D- RSP):

The purpose here is to give a complete prescription for determine the energy level of n^{th} excited states, of hydrogenic atoms with *NCPLHRT*, we first find the corrections $E_{\text{u-clh}}(n,\lambda,g,Z)$ and $E_{\text{d-clh}}(n,\lambda,g,Z)$ for hydrogenic atoms which have $j=l+1/2$ (spin up) and $j=l-1/2$ (spin down), respectively, at first order of two parameters (Θ and $\bar{\theta}$) obtained by applying the standard perturbation theory to find the following:

$$\begin{aligned}E_{u-\mathrm{clh}}(n,\lambda,g,Z)&=-\gamma\left(N\frac{\mathrm{n}!\Gamma(l+3/2)}{\Gamma(l+n+3/2)}\right)^2k_+\int_0^{+\infty}r^{2l+2}\exp(-\lambda r^2)\left[L_n^{l+1/2}(\lambda r^2)\right]^2\left(\left(\frac{Z}{2r^3}+\frac{g}{2r}+\lambda\right)\Theta-\frac{\bar{\theta}}{2}\right)dr\\ E_{d-\mathrm{clh}}(n,\lambda,g,Z)&=-\gamma\left(N\frac{\mathrm{n}!\Gamma(l+3/2)}{\Gamma(l+n+3/2)}\right)^2k_-\int_0^{+\infty}r^{2l+2}\exp(-\lambda r^2)\left[L_n^{l+1/2}(\lambda r^2)\right]^2\left(\left(\frac{Z}{2r^3}+\frac{g}{2r}+\lambda\right)\Theta-\frac{\bar{\theta}}{2}\right)dr\end{aligned} \tag{27.1}$$

Now, we can write the above two equations to the new form:

$$\begin{aligned}E_{u-\mathrm{clh}}(n,\lambda,g,Z)&=-\gamma\left(N\frac{\mathrm{n}!\Gamma(l+3/2)}{\Gamma(l+n+3/2)}\right)^2k_+\left\{\Theta\sum_{i=1}^{3}T_i(n,\lambda,g,Z)+\frac{\bar{\theta}}{2\mu}T_4(n,\lambda,g,Z)\right\}\\ E_{d-\mathrm{clh}}(n,\lambda,g,Z)&=-\gamma\left(N\frac{\mathrm{n}!\Gamma(l+3/2)}{\Gamma(l+n+3/2)}\right)^2k_-\left\{\Theta\sum_{i=1}^{3}T_i(n,\lambda,g,Z)+\frac{\bar{\theta}}{2\mu}T_4(n,\lambda,g,Z)\right\}\end{aligned} \tag{27.2}$$

Moreover, the expressions of the four factors $T_i(n,\lambda,g,Z)$ ($i=\overline{1,4}$) are given by:

$$\begin{aligned}T_1(n,\lambda,g,Z)&=\frac{Z}{2}\int_0^{+\infty}r^{2l-1}\exp(-\lambda r^2)\left[L_n^{l+1/2}(\lambda r^2)\right]^2dr\\ T_2(n,\lambda,g,Z)&=\frac{g}{2}\int_0^{+\infty}r^{(2l+2)-1}\exp(-\lambda r^2)\left[L_n^{l+1/2}(\lambda r^2)\right]^2dr\\ T_3(n,\lambda,g,Z)&=-\lambda T_4(n,\lambda,g,Z)\\ &=\lambda\int_0^{+\infty}r^{(2l+3)-1}\exp(-\lambda r^2)\left[L_n^{l+1/2}(\lambda r^2)\right]^2dr\end{aligned} \tag{28}$$

To evaluate the above four factors, which depended with the parameters of potential (λ,g,Z), we apply the following special integration, the results of eq. (29) (see below) depended with the generalized hypergeometric function ${}_3F_2(-m,\varepsilon,\varepsilon-\beta;-n+\varepsilon,\lambda+1;1)$ which obtained from the generalized function ${}_pF_q(\alpha_1,...,\alpha_p,\beta_1,....,\beta_q,z)$ for $p=3$ and $q=2$, in addition to the usual Gamma function $\Gamma(x)$. After straightforward calculations, we can obtain the explicitly results [23]:

$$\int_0^{+\infty}t^{\varepsilon-1}\exp(-\omega t)L_m^{\lambda}(\omega t)L_n^{\beta}(\omega t)dt=\frac{\omega^{-\varepsilon}\Gamma(n-\varepsilon+\beta+1)\Gamma(m+\lambda+1)}{m!n!\Gamma(1-\varepsilon+\beta)\Gamma(1+\lambda)}\,{}_3F_2(-m,\varepsilon,\varepsilon-\beta;-n+\varepsilon,\lambda+1;1) \tag{29}$$

Thus, it is easily to obtain the explicitly results:

$$\begin{aligned}
T_1(n,\lambda,g,Z) &= \frac{Z}{2}\frac{\lambda^{-2l}\Gamma(n-l+3/2)\Gamma(n+l+3/2)}{n!^2\Gamma(3/2-l)\Gamma(l+3/2)}\,{}_3F_2(-n,2l,l-1/2;-n+2l,l+3/2;1)\\
T_2(n,\lambda,g,Z) &= \frac{g}{2}\frac{\lambda^{-(2l+2)}\Gamma(n-l-1/2)\Gamma(n+l+3/2)}{n!^2\Gamma(-l-1/2)\Gamma(l+3/2)}\,{}_3F_2(-n,2l+2,l+3/2;-n+2l+2,l+3/2;1)\\
T_3(n,\lambda,g,Z) &= -\lambda T_4(n,\lambda,g,Z)\\
&= \lambda\frac{\lambda^{-(2l+3)}\Gamma(n-l-3/2)\Gamma(n+l+3/2)}{n!^2\Gamma(-l-3/2)\Gamma(l+3/2)}\,{}_3F_2(-n,2l+3,l+5/2;-n+2l+3,l+3/2;1)
\end{aligned} \tag{30}$$

It is well known that $l=\{0,1...,n-1\}\geq 0$, then ($(-l-1/2)$ and $(-l-3/2)$) are representing two negative values and we having $\Gamma(-l-1/2)=\Gamma(-l-3/2)=+\infty$, which gives $T_2(n,\lambda,g,Z)=T_3(n,\lambda,g,Z)=T_4(n,\lambda,g,Z)=0$, allowed us to obtain the exact modifications $E_{\text{u-clh}}(n,\lambda,g,Z)$ and $E_{\text{d-clh}}(n,\lambda,g,Z)$ of n^{th} excited states of hydrogenic atoms with *NCPLHRT*, which produced by modified spin-orbital effect $H_{so-clh}(r,\Theta,\bar{\theta})$ as:

$$\begin{aligned}
E_{u\text{-clh}}(n,\lambda,g,Z) &= -\gamma N^2\Theta\frac{Z}{2}\left(\frac{\Gamma(l+3/2)\Gamma(n-l+3/2)}{\Gamma(3/2-l)\Gamma(l+n+3/2)}\right)k_+\lambda^{-2l}\,{}_3F_2(-n,2l,l-1/2;-n+2l,l+3/2;1)\\
E_{d\text{-clh}}(n,\lambda,g,Z) &= -\gamma N^2\Theta\frac{Z}{2}\left(\frac{\Gamma(l+3/2)\Gamma(n-l+3/2)}{\Gamma(3/2-l)\Gamma(l+n+3/2)}\right)k_-\lambda^{-2l}\,{}_3F_2(-n,2l,l-1/2;-n+2l,l+3/2;1)
\end{aligned} \tag{31}$$

Thus, our research also conveys another innovative feature exact, the extended global quantum group symmetry (NC: 3D-RSP) is broken simultaneously and replaced by new quantum subgroup symmetry (NC: 3D-RS) under interaction of hydrogenic atoms with *NCPLHRT*.

3.3 The Exact Modified Magnetic Spectrum of Hydrogenic Atoms Under NCPLHRT Interactions in Residual Group (NC: 3D- RS):

Further to the important previously obtained results, now, we consider another physically meaningful phenomena produced by the effect of *NCPLHRT* related to the influence of an external uniform magnetic field $\vec{B}$, to avoid the repetition in the theoretical calculations, it's sufficient to apply the following replacements:

$$\begin{cases}\vec{\Theta}\to\chi\vec{B}\Rightarrow-\left(\dfrac{Z}{2r^3}+\dfrac{g}{2r}+\lambda\right)\Theta\to-\left(\dfrac{Z}{2r^3}+\dfrac{g}{2r}+\lambda\right)\vec{B}\vec{L}\\ \vec{\theta}\to\bar{\sigma}\vec{B}\equiv\bar{\sigma}B\Rightarrow\dfrac{\bar{\theta}}{2}\to\dfrac{\bar{\sigma}}{2}\vec{B}\vec{L}\end{cases} \tag{32}$$

Here χ and $\vec{\sigma}=\bar{\sigma}\vec{k}$ are two infinitesimal real proportional's constants, and we choose the arbitrary external magnetic field $\vec{B}=B\vec{k}$ parallel to the (Oz) axis, which allow us to introduce the new modified magnetic Hamiltonian H_{m-clh} in (NC: 3D-RSP) symmetries as:

$$H_{m-clh}=-\left\{\left(\frac{Z}{2r^3}+\frac{g}{2r}+\lambda\right)\chi-\frac{\bar{\sigma}}{2}\right\}\aleph_{\text{mod}-z} \tag{33}$$

Here $\aleph_{\text{mod}-z}\equiv\vec{B}\vec{J}-\aleph_z$ denote to the modified Zeeman effect while $\aleph_z\equiv-\vec{S}\vec{B}$ is the ordinary Hamiltonian operator of Zeeman Effect. To obtain the exact noncommutative magnetic modifications of energy $E_{\text{mag-clh}}(n,m,\alpha)$, we just replace k_+ and Θ into the eq. (31) by the following parameters: m and χ, respectively:

$$E_{mag-clh}(n,m,\lambda,g,Z)=-\gamma N^2\Theta\frac{Z}{2}\left(\frac{\Gamma(l+3/2)\Gamma(n-l+3/2)}{\Gamma(3/2-l)\Gamma(l+n+3/2)}\right)\chi\lambda^{-2l}\,{}_3F_2(-n,2l,l-1/2;-n+2l,l+3/2;1)Bm \tag{34}$$

We have $-l\leq m\leq +l$, which allow us to fixing $(2l+1)$ values for discreet number m.

4. RESULTS AND DISCUSSION

We are now in a position to attack the main objective of our study, let us resume the nonrelativistic modified eigenenergies $E_{\text{nc-u}clh}(n,j,l,s,m,\alpha)$ and $E_{\text{nc-d}clh}(n,j,l,s,m,\alpha)$ of a hydrogenic atoms under *NCPLHRT* obtained from solving MSE for n^{th} excited states in (NC: 3D-RSP) symmetries. On based to our original results presented on the Eqs. (31) and (34), in addition to the ordinary energy E_{nl} for *NCPLHRT*, which presented in the eq. (8):

$$E_{\text{nc -uclh}}\left(n,\lambda,g,Z,j,l,s,m\right)=\sqrt{2\lambda}\left(2n+l+3/2\right)+\sum_{j=0}^{\infty}\left(b_j^{(0)}\lambda^{j+1/2}+b_j^{(1)}\lambda^{j+1/4}+b_j^{(2)}\lambda^{j}+b_j^{(3)}\lambda^{-(j+1/4)}\right)$$
$$-\gamma N^2\Theta\frac{Z}{2}\left(\frac{\Gamma(l+3/2)\Gamma(n-l+3/2)}{\Gamma(3/2-l)\Gamma(l+n+3/2)}\right)\lambda^{-2l}{}_3F_2\left(-n,2l,l-1/2;-n+2l,l+3/2;1\right)\left(\Theta k_{+}+Bm\chi\right) \tag{35}$$

and

$$E_{\text{nc -dclh}}\left(n,\lambda,g,Z,j,l,s,m\right)=\sqrt{2\lambda}\left(2n+l+3/2\right)+\sum_{j=0}^{\infty}\left(b_j^{(0)}\lambda^{j+1/2}+b_j^{(1)}\lambda^{j+1/4}+b_j^{(2)}\lambda^{j}+b_j^{(3)}\lambda^{-(j+1/4)}\right)+$$
$$-\gamma N^2\Theta\frac{Z}{2}\left(\frac{\Gamma(l+3/2)\Gamma(n-l+3/2)}{\Gamma(3/2-l)\Gamma(l+n+3/2)}\right)\lambda^{-2l}{}_3F_2\left(-n,2l,l-1/2;-n+2l,l+3/2;1\right)\left(\Theta k_{-}+Bm\chi\right) \tag{36}$$

This is the main goal of this work, It's clearly, that the obtained eigenvalues of energies are real's and then the NC diagonal Hamiltonian H_{nc-clh} is Hermitian, furthermore it's possible to writing the three elements: $\left(H_{nc-\text{clh}}\right)_{11}$, $\left(H_{nc-\text{clh}}\right)_{22}$ and $\left(H_{nc-\text{clh}}\right)_{33}$ of NC nonrelativistic Hamiltonian describing hydrogenic atoms with *NCPLHRT* as follows:

$$\begin{aligned}\left(H_{nc-\text{clh}}\right)_{11}&=-\frac{\Delta_{nc}}{2}+H_{\text{int}-u\text{clh}}\\ \left(H_{nc-\text{clh}}\right)_{22}&=-\frac{\Delta_{nc}}{2}+H_{\text{int}-d\text{clh}}\\ \left(H_{nc-\text{clh}}\right)_{33}&=-\frac{\Delta}{2}-Z/r+gr+\lambda r^2\end{aligned} \tag{37}$$

Where the new kinetic energy $\frac{\Delta_{nc}}{2}$ and the two modified interactions $H_{\text{int}-uclh}$ and $H_{\text{int}-dclh}$ are given by:

$$\begin{aligned}\frac{\Delta_{nc}}{2}&=\frac{\Delta-\vec{\vec{\theta}}\vec{L}-\vec{\vec{\sigma}}\vec{L}}{2}\\ H_{\text{int}-u\text{clh}}&=-Z/r+gr+\lambda r^2-\gamma\left(k_{+}\Theta+\chi\aleph_{\text{mod}-z}\right)\left(\frac{Z}{2r^3}+\frac{g}{2r}+\lambda\right)\\ H_{\text{int}-d\text{clh}}&=-Z/r+gr+\lambda r^2-\gamma\left(k_{-}\Theta+\chi\aleph_{\text{mod}-z}\right)\left(\frac{Z}{2r^3}+\frac{g}{2r}+\lambda\right)\end{aligned} \tag{38}$$

Thus, the ordinary kinetic term for *CPLHRT* $\left(-\frac{\Delta}{2}\right)$ and ordinary interaction ($-Z/r+gr+\lambda r^2$) are replaced by new modified form of kinetic term $\left(-\frac{\Delta_{nc}}{2}\right)$ and new modified interactions ($H_{\text{int}-u\text{clh}}$ and $H_{\text{int}-d\text{clh}}$). On the other hand, it is evident to consider the quantum number m takes ($2l+1$) values and we have also two values for $j=l\pm\frac{1}{2}$, thus every state in usually three-dimensional space of energy for new *NCPLHRT* will be $2(2l+1)$ sub-states. To obtain the total complete degeneracy of energy level of *NCPLHRT* in NC 3-dimension spaces-phases, we need to sum for all allowed values of l. Total degeneracy is thus,

$$2\sum_{i=0}^{n-1}\left(2l+1\right)\equiv 2n^2 \tag{39}$$

Note that the obtained new energy eigenvalues ($E_{\text{nc -uclh}}\left(n,\lambda,g,Z,j,l,s,m\right)$ and $E_{\text{nc -dclh}}\left(n,\lambda,g,Z,j,l,s,m\right)$) depend to new discrete atomic quantum numbers $\left(n,j,l,s\right)$ and m in addition to the parameters (λ,g,Z) of the *NCPLHRT*. Paying attention to the behaviour of the spectrums (35) and (36) ($E_{\text{nc -uclh}}\left(n,\lambda,g,Z,j,l,s,m\right)$ and $E_{\text{nc -dclh}}\left(n,\lambda,g,Z,j,l,s,m\right)$), it is possible to recover the results of commutative space when we consider $\left(\Theta,\chi\right)\to\left(0,0\right)$.

5. CONCLUSION

In this paper three-dimensional MSE for *NCPLHRT* has been solved via Bopp's shift method and independent time standard perturbation theory in (NC: 3D-RSP) symmetries, we resume the main obtained results:

1) The exact energy spectrum ($E_{\text{nc -uclh}}\left(n,\lambda,g,Z,j,l,s,m\right)$ and $E_{\text{nc -dclh}}\left(n,\lambda,g,Z,j,l,s,m\right)$) for n^{th} excited levels, for hydrogenic atoms,

2) Ordinary interaction ($-Z/r+gr+\lambda r^2$) were replaced by *NCPLHRT* ($H_{\text{int}-uclh}$ and $H_{\text{int}-dclh}$) for hydrogenic atoms,

3) The ordinary kinetic term $-\frac{\Delta}{2}$ modified to the new form $\frac{\Delta_{nc}}{2}=\frac{\Delta-\vec{\vec{\theta}}\vec{L}-\vec{\vec{\sigma}}\vec{L}}{2}$ for *NCPLHRT*,

4) We have shown that, the group symmetry (NC: 3D-RSP) corresponding *NCPLHRT* were broken simultaneously and replaced by the new residual symmetry sub-group (NC: 3D-RS).

5) It has been shown that, the MSE presents useful

rich spectrums for improved understanding of hydrogenic atoms influenced by the *NCPLHRT* and we have seen also that the modified of spin-orbital and modified Zeeman effect were appears du the presence of the two infinitesimal parameters (Θ, χ) which are induced by position-position noncommutativity property of space.

AKNOWLEDGEMENTS

Author is grateful to the Editor-in-Chief of the Journal of Nano- and Electronic Physics Protsenko Ivan Yuhymovych for a critical reading of the manuscript and his valuable comments. This work was supported with search laboratory of: Physics and Material Chemistry, in Physics department, Sciences faculty, University of M'sila, Algeria, The author would like to thank the kind's referees for the positive comments and suggestions, which led to the improved standard of this article.

REFERENCES

1. Enrique Castro, Pablo Martin, *J. Phys. A: Math. Gen.* **33**, 5321 (2000).
2. Tapas Das, *EJTP* **13**, No **35**, 207 (2016).
3. J. Killingbeck, *Phys. Lett. A* **65**, 87 (1978).
4. Abdelmadjid Maireche, *Afr. Rev. Phys.* **9**, 0060 (2014).
5. A. Connes, M.R. Douglas, A. Schwarz, *JHEP* **9802**, 003 (1998).
6. Abdelmadjid Maireche, *Afr. Rev Phys.* **11**, 111 (2016).
7. A.E.F. Djemei, H. Smail, *Commun. Theor. Phys.* **41**, 837 (2004).
8. Joohan Lee, *J. Korean Phys. Soc.* **47** No 4, 571 (2005).
9. Abdelmadjid Maireche, *NanoWorld J.* **1** No 4, 122 (2016).
10. Li Fateme Hoseini, Jayanta K. Saha, Hassan Hassanabadi, *Commun. Theor. Phys.* **65**, 695 (2016).
11. O. Bertolami, P. Leal, *Phys. Lett. B* **750**, 6 (2015).
12. Abdelmadjid Maireche, *J. Nano- Electron. Phys.* **7** No 4, 04021 (2015).
13. C. Bastos, O. Bertolami, N.C. Dias, J.N. Prata, *J. Math. Phys.* **49**, 072101 (2008).
14. Abdelmadjid Maireche, *Med. J. Model. Simul.* **04**, 060 (2015).
15. Shi-Hai Dong, *Wave Equations in Higher Dimensions* (Springer Science+Business Media B.V.: 2011).
16. J. Gamboa, M. Loewe, J.C. Rojas, *Phys. Rev. D* **64**, 067901 (2001).
17. M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, *Phys. Rev. Lett.* **86**, 2716 (2001).
18. Abdelmadjid Mairechee, *International Letters of Chemistry, Physics and Astronomy* **60**, 11 (2015).
19. Abdelmadjid Maireche, *International Frontier Science Letters* **9**, 33 (2016).
20. Abdelmadjid Maireche, *Afr. Rev. Phys.* **11**, 175 (2016).
21. Abdelmadjid Maireche, *J. Nano- Electron. Phys.* **8** No 2, 02027 (2016).
22. Abdelmadjid Maireche, *J. Nano- Electron. Phys.* **8** No 2, 02046 (2016).
23. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Dover Publications* (New York: 1965).